The sweet spot of a baseball bat

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The physics of the three sweet spots of a baseball bat is discussed and the location of the ball impact point on the bat that leads to maximum "power" (greatest batted ball speed) is determined.

I. INTRODUCTION

A recent article in this Journal[1] explained the physics of the three sweet spots of a tennis racket (the center of percussion, the node of the first harmonic, and the maximum of the coefficient of restitution). This article will show that for a baseball bat, two of these sweet spots (COP and node) are similar in nature to those found in a tennis racket, but, unlike the tennis racket, the maximum in the COR is a very weak function of the position of where the ball hits the bat and therefore, the COR can be considered constant along the bat. However, there is an optimum point along the bat to hit the ball to maximize the ball's velocity after the collision. The location of this point can be calculated from the dynamics of the collision and it is a function of the ball and bat velocities before the impact, the mass of the ball, and the mass and moment of inertia of the bat.

II. THE CENTER OF PERCUSSION

A bat of mass M and with initial velocity zero can be treated as a free-body that is hit by a ball whose momentum changes by Δp due to the interaction. The ball hits the bat at a distance b from the bat center of mass (CM) and the bat is normally held at a distance a from the CM. To conserve linear momentum, Δp must equal $M * V$, where V is the final velocity of the bat CM. To conserve angular momentum (about the CM) $b * \Delta p$ must equal $I * wf$, where I is the bat moment of inertia (about the CM) and wf is the angular velocity of the bat after the interaction. Because the direction of the velocity of the bat handle due to the rotation about the CM is in the opposite direction to the direction of the velocity of the CM, it is possible, at some point, for the two to cancel, and there be no net translational motion at that point. It is desirable (to a batter) to have that point be the one where the hands grip the bat, since that will minimize the initial shock or jar to the hitter. If the batter holds the bat a distance a from the bat CM, the condition for this cancellation to take place at the hands is $V = wf * a$. The distance b which will make a the effective pivot point of the bat can be calculated by solving the above three conditions; the result is

$$b = I / M * a. \quad (1)$$

The value of the moment of inertia of a bat about its CM can be measured quite easily with a torsion pendulum,[2] and a typical value (for an aluminum softball bat) is 0.046 kg m^2. A second, but similar, method to find the COP point on the bat is to pivot the bat from the handle as a physical pendulum and measure the period of its oscillation. This actually measures the moment of inertia above the pivot point, and then the parallel axis theorem can be used to get the moment about the CM which then can be used in Eq. (1).

For an aluminum softball bat of length 0.81 m and mass 0.825 kg, b came out to be 0.18 m beyond the CM (0.17 m from the fat or distal end of the bat).

III. THE NODE

When you hit the ball in the wrong place on the bat and your hands "sting," it means that you have excited a higher harmonic of the bat's natural oscillation. If you hit the ball at the location of the node of this oscillation, you will not excite it, your hands will not sting, and you will have hit the ball at the second sweet spot. For a uniform beam, this node is approximately $\frac{1}{4}$ of the length of the beam from the end. To find the node, hold the bat by two fingers, about 0.15–0.20 m from the handle end, and strike the bat at various locations along its length. The bat will sing out or resonate when it is hit unless you hit it at the node. The amplitude of the oscillation (the loudness of the sound) will increase the further away from the node the bat is struck.

The frequency of this oscillation was measured using a microphone, audio amplifier, and an oscilloscope. For the aluminum bat used in these experiments, the node was 0.19 m from the end of the bat (0.16 m from the CM) and the frequency of the oscillation was 220 Hz.

IV. THE LOCATION OF THE MAXIMUM OF THE COEFFICIENT OF RESTITUTION

When a tennis racket's handle is firmly clamped in a vise, the COR varies from 0.65 near the throat to 0.2 or lower near the tip of the racket.[3] This is because the racket frame is flexible and any energy that goes into racket deformation is lost.[4] A similar argument can be made for a baseball bat, where any energy that goes into the deformation of the bat is lost since by the time the bat "springs back," the ball has departed. (The ball spends 1 ms in contact with the bat[5] which is much less than the half-period of the bat's oscillation.) However, a baseball bat is 7 to 15 times as stiff as a tennis racket, and when it is hit by the ball, it deforms much less. Therefore, the amount of energy that goes into the bat deformation is very small compared to the very large direct energy lost in the ball–bat collision. If this is the case, then the variation of this deformation energy loss with position along the bat can be neglected, and the COR of the bat can be considered essentially constant.

Is there then an optimum location along the bat to hit the ball to maximize the "power"—to maximize the rebound velocity of the struck ball if the COR is considered to be constant? In the literature it is said that there is such a point and it is located at the center of percussion.[6,7] In Sec. V the maximum power point will be found and it will be shown that it is not at the COP.

V. THE MAXIMUM "POWER" POINT

Since the ball only spends 1 ms in contact with the bat, it is probably better to treat the bat as a free-body, rather than an object clamped at one end.[8] Working in a frame of reference moving with the bat CM, it is possible to calculate the rebound velocity of a ball that is struck. (Note this frame of reference will continue to move after the ball–bat interaction, but it will no longer be at the bat CM.) This rebound velocity of the ball will be a function of the location on the bat where the ball hits, and it can be maximized with respect to this parameter, yielding the optimum location. To simplify the problem, it will be assumed that the ball hits and rebounds normal to the bat and is in the plane of the swing, reducing this to a one-dimensional problem, even though the bat is allowed to have a rotation about an axis perpendicular to the plane of the swing.

The three equations below come from conservation of linear momentum, conservation of angular momentum, and the definition of coefficient of restitution.[9] The angular velocity w is taken about the bat CM and clockwise is defined as positive, as shown in Fig. 1. This is all in the frame of reference moving with the initial velocity of the CM of the bat and the bat is considered a free-body.

$$m^*vi = m^*vf + M^*Vf,$$

$$b^*m^*vi + I^*wi = b^*m^*vf + I^*wf,$$

$$e^*(vi - wi^*b) = Vf + wf^*b - vf,$$

where m and M are the ball and bat mass, v and V are the ball and bat velocity (initial and final), w is the bat angular velocity (initial and final), and e is the coefficient of restitution.

Solving these three equations for vf by eliminating Vf and wf gives

$$vf = vi - \{(1+e)^*(vi - wi^*b)/[1 + m/M + (m^*b^2/I)]\}.$$

Note that when $wi = 0$, this expression is a function of b^2, so it must have an extremum (maximum in this case) at $b = 0$, the center of mass of the bat. This is, of course, expected, since if the bat has no initial rotational energy, a hit at the CM will give it no final rotational energy.

To find the value of b which maximizes vf in the general case, the expression for vf is differentiated with respect to b and the result set equal to zero.

$$wi^*b^2 - 2^*vi^*b - [wi^*(M+m)^*I]/(m^*M) = 0.$$

This expression is independent of e, the COR, and can be solved for b:

$$b = vi/wi \pm \sqrt{[(vi/wi)^2 + I^*(m+M)/(M^*m)]}.$$

It is quite clear that this point is not the COP location, because the position is a function of the ball and bat velocity in addition to the properties of the bat. Since vi/wi is negative, the + sign is to be used here on the square root. The values of m, M, and I are easy to obtain in the lab, but the values of wi and vi are determined by the hitter swinging the bat and the pitcher throwing the ball.

The value of b for a range of these parameters can be obtained, but it is instructive to first see how b varies with each of these variables. If

$$I^*(m+M)^*wi^2/(M^*m)^*vi^2$$

is less than one, the square root can be expanded and the result is

$$b = [(r+1)^*wi^*k^2]/vi,$$

where $r = M/m$ and I has been replaced by M^*k^2.

This means that b, the distance from the CM of the bat to the impact point of the ball, increases as the mass of the bat increases, increases as the moment of inertia of the bat increases, and increases the more "wristy" is the swing. The value of b decreases the faster the pitch is thrown and the more it is a body–arm swing as opposed to a wrist swing. Since most players rarely change their style of swing, the main things learned from this exercise are:

(i) You should hit a fast pitch closer to your hands for maximum power and you should try to hit a slow pitch further out on your bat to get best results.

(ii) With a larger bat/ball mass ratio (hardball) hit the ball out further from the CM, and with a smaller bat/ball mass ratio (softball) hit it closer to your hands.

Table I gives the exact value of b (the contact distance from the CM) that maximizes the outgoing ball speed, for several bat/ball mass ratios and a number of vi/wi ratios. The ratio of vi (the ball speed in a frame of reference moving with the original bat CM velocity) divided by wi (the angular velocity of the bat at contact) is a length and it is convenient to use the bat length L as a measure of this ratio. The results listed in Table I and displayed in Fig. 2 show that to obtain maximum batted ball speed, the ball should be hit somewhat closer to your hands than the COP or the node point. As an example of this, if the bat's effective pivot point during the swing was between the wrist and the batter's body (a distance L from the bat CM), and the pitched ball and the bat CM had the same linear speed in the home plate frame of reference, the vi/wi ratio would be 2^*L. If the bat/ball mass ratio were 6, the predicted location to hit the

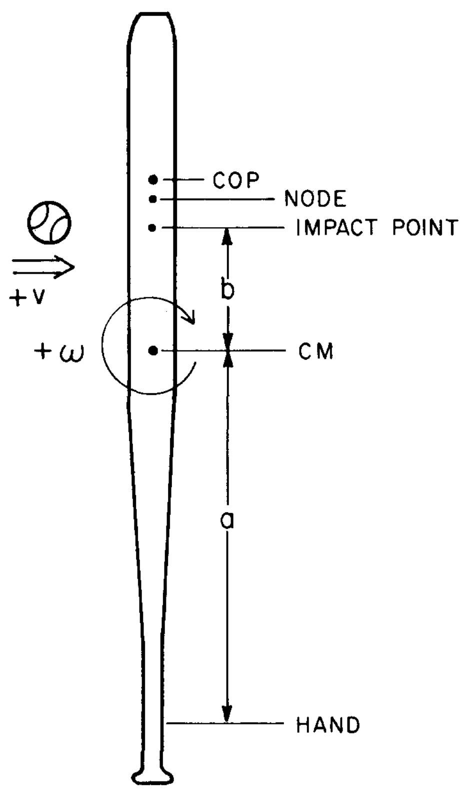

Fig. 1. Location of several points of interest on a baseball bat.

Table I. The distance from the maximum power point to the CM of the bat is given for several values of the bat/ball mass ratio and the ball speed (in the bat frame of reference) divided by the angular velocity of the bat, in units of the bat length. This is for an aluminum softball bat of mass 0.825 kg, moment of inertia about the CM of 0.046 kg m^2, and length 0.81 m.

Bat/ball mass ratio	Linear speed/ angular speed	Distance from bat CM in inches	Distance from bat CM in meters
5.000	1.000	7.192	0.182
5.000	1.250	5.956	0.151
5.000	1.500	5.066	0.128
5.000	1.750	4.399	0.111
5.000	2.000	3.882	0.098
5.000	2.250	3.472	0.088
5.000	2.500	3.138	0.079
5.000	2.750	2.863	0.072
5.000	3.000	2.631	0.067
6.000	1.000	8.266	0.209
6.000	1.250	6.876	0.174
6.000	1.500	5.864	0.148
6.000	1.750	5.101	0.129
6.000	2.000	4.508	0.114
6.000	2.250	4.035	0.102
6.000	2.500	3.650	0.092
6.000	2.750	3.331	0.084
6.000	3.000	3.062	0.078
7.000	1.000	9.312	0.236
7.000	1.250	7.777	0.197
7.000	1.500	6.650	0.168
7.000	1.750	5.795	0.147
7.000	2.000	5.128	0.130
7.000	2.250	4.594	0.116
7.000	2.500	4.159	0.105
7.000	2.750	3.797	0.096
7.000	3.000	3.492	0.088

ball would be 0.114 m beyond the bat CM, while the COP and node to CM distances are 0.18 and 0.16 m, respectively. For "fungo" hitting (ball is at rest in the home plate frame), the point would be out beyond the COP or node,

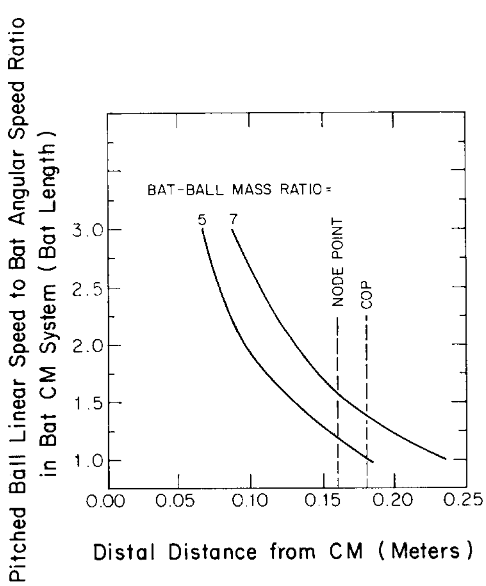

Fig. 2. Location of the ball impact point on a bat that will produce maximum "power" (batted ball speed).

since the vi/wi ratio would be approximately L. However, for this type of hitting, a lighter bat with a different weight distribution is normally used.[8,10]

If there is any appreciable energy loss due to bat deformation, the point of maximum power will move outward toward the node if the bat acts as if it were a free-body (which is what is assumed in this article) and the point will move toward the hands if the bat acts as if it were clamped firmly at the handle.

Throughout this article, SI units have been used, with the exception of the final column for b in Table I, where both inches and SI were employed, since the official rules of baseball are specified in English units.

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I would like to thank Professor Peter Brancazio for getting me interested in this problem with private communications and his book *SportScience*.[10]

APPENDIX

The value of b that maximizes vf is independent of e, the coefficient of restitution. If $e = 1$, the collision is elastic and the final kinetic energy of the ball and the bat will equal the initial kinetic energy, including the rotational energy of the bat ($\frac{1}{2}*I*wi^2$). For such a collision, the value of vf should maximize when the bat has no final rotational energy ($wf = 0$) and it has only translational energy. But what happens in an inelastic collision? Can wf be set equal to zero to maximize vf? Taking the original three equations, letting $wf = 0$, and solving for b gives

$$b = vi^*(1+e)/2^*wi \pm (\{[vi^*(1+e)/2^*wi]^2 + [I^*(m+M)^*wi^2]/(e^*M^*m)\}^{1/2}).$$

This reduces to the previously obtained value of b only for elastic collisions ($e = 1$), so the condition that vf be a maximum leads to no rotational energy left in the bat only for elastic collisions.

[1]H. Brody, Am. J. Phys. **49**, 816 (1981).
[2]H. Brody, Phys. Teach. **23**, 213 (1985).
[3]What is actually measured is not the COR, but the ratio of ball rebound velocity to ball incident velocity. They will be the same to the extent that the racket (or bat) recoil velocity can be neglected.
[4]H. Brody, Am. J. Phys. **47**, 482 (1979).
[5]S. Plagenhoef, *Patterns of Human Motion* (Prentice-Hall, Englewood Cliffs, NJ, 1971), p. 59. This result was based on 4000-frame/s photography and showed a 1.25-ms contact time (five frames).
[6]W. C. Connolly and W. Christian, Phys. Ed. **37**, 21 (1980); F. O. Bryant, L. N. Burkett, S. S. Chen, G. S. Krahenbuhl, and P. Lu, Res. Q. **48**, 231 (1977).
[7]H. Head, US Patent 3999756, 28 December 1976.
[8]P. Kirpatrick [Am. J. Phys. **31**, 606 (1963)] showed that if the batter could put energy into the bat during the time that the ball was in contact with the bat, the location to hit the ball was $b = I^*w/M^*V$ if maximum power was desired. Since the ball spends only 1 ms in contact with the bat, Kirkpatrick notes that this solution to the problem is not realistic.
[9]The COR e is the ratio of the relative velocity of the ball and bat contact point after the collision to their relative velocity before the collision. The idea of using the COR in collisions was first published (in Latin) by both J. Wallis and C. Wren in Philos. Trans. **43**, 864 and 867 (1668).
[10]P. Brancazio, *SportScience* (Simon and Schuster, New York, 1984).

A class of exactly soluble many-body Hamiltonians

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A class of exactly soluble N-particle Hamiltonians is presented. The matrix representation of these Hamiltonians, which consists of one- and two-body interactions, is block diagonal. The blocks consist of a finite-dimensional matrices which can be easily diagonalized. An example for a system with two particles is discussed.

I. INTRODUCTION

The discussion of the solution of the many-body Schrödinger equation is one of the thorniest aspects of the teaching of introductory quantum mechanics. Part of the problem is due to the lack of exactly soluble Hamiltonians with two-body interactions. Thus one is usually forced to introduce fundamental concepts together with the explanation of the particular details of approximate methods, such as the Hartree–Fock formalism. This tends to cause confusion on the part of the students, who very often get lost in the mathematical manipulations.

To separate the fundamental concepts from the discussion of the mathematical details of a particular method of approximation it would be advantageous to use Hamiltonians which are exactly soluble. In addition, one would hope that such systems contain two-body interactions but are relatively easy to solve. In this paper we present one class of many-body Hamiltonians which conforms to these characteristics. They consist of one- and two-particle interactions, where the latter have been chosen to render the problem easily solvable. However, they retain enough complexity to be effective in the explanation of many of the fundamental properties of N-particle systems, such as the energy splittings between different total spin eigenstates. These Hamiltonians lead to exact solutions because their matrix representations, in the appropriate basis sets, turn out to be block diagonal, where the blocks consist of finite-dimensional submatrices.

The paper is organized as follows. In Sec. II we postulate these Hamiltonians and discuss their general form. In Sec. III we consider an example consisting of a system with two particles. The conclusions are presented in Sec. IV.

II. GENERAL FORM OF THE HAMILTONIANS

Let us consider a system consisting of N particles which, for definiteness, we will take to be spin-$\frac{1}{2}$ fermions. In general, a Hamiltonian consisting of one- and two-body interactions can be written in the form

$$\hat{H} = \sum_{\mu=1}^{N} \hat{h}(\mu) + \sum_{\mu \neq \nu}^{N} \hat{g}(\mu,\nu) , \tag{1}$$

where μ and ν run over the coordinates of the N particles. Here, $\hat{h}$ is the one-particle Hamiltonian, whereas $\hat{g}$ corresponds to the two-body interaction.

Consider the set of single-particle states $|\phi_n(\mu)\rangle$ which is complete and orthonormal, consisting of the eigenstates of the one-particle Hamiltonian. In other words, we have

$$\hat{h}(\mu)|\phi_n(\mu)\rangle = \epsilon_n |\phi_n(\mu)\rangle ,$$

where the ϵ_n are the single-particle energies. With these single-particle states we can construct a complete basis for the N-particle Hilbert space. For convenience, we will use the basis obtained by taking Cartesian products[1] of the $|\phi_n(\mu)\rangle$, i.e., the N-particle basis functions have the form $|\phi_{n_1}(1)\phi_{n_2}(2)\cdots\phi_{n_N}(N)\rangle$. Therefore, any admissible N-

