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Performance versus moment of inertia of sporting implements

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A review is presented of the importance of the moment of inertia (MOI) to the performance of a sporting instrument. It is shown that for a given coefficient of restitution (COR), both the intrinsic power and the swing speed of a tennis racquet or baseball bat correlate strongly with the MOI about an axis through the handle and only weakly with the mass. It is further shown that for non-wood baseball bats approved for use in college baseball in the USA, batted ball speed is a stronger function of COR than of MOI. When comparing two implements with the same MOI, the primary factor affecting performance is the COR, which can be enhanced by means of the trampoline effect. A new method of matching the MOI of a set of golf clubs is also described. © 2009 John Wiley and Sons Asia Pte Ltd

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1. INTRODUCTION

The physics of sporting implements has been considered in journal articles [1–4] and in several books on the subject [5–9]. There is one particular aspect that has not received as much attention as it deserves, despite the fact that it is foremost on the minds of many players. That is, how does the performance of a sporting implement depend on its physical properties? A common view is that heavy implements are more powerful than light instruments. Moreover, heavy instruments cannot be swung as fast as light instruments. Both statements are technically incorrect. Most implements used in any particular sport tend to be similar in weight, in which case the common view can be both misleading and of no help when comparing implements of the same weight. The primary factor determining both power and swing speed, regardless of the mass of the implement, is its moment of inertia (MOI). In this review, it is shown why this is the case, giving specific examples from baseball, tennis, and golf.

The MOI of a sporting implement is not as well-defined as its mass or length since it depends on the arbitrary axis chosen to measure it. Moreover, the chosen axis is not necessarily the one chosen by a player to swing it. For practical convenience and consistency, the MOI of a baseball bat is conventionally measured by swinging the bat about an axis within the handle located 15 cm (6 inches) from the knob end. Similarly, the MOI of a tennis racquet is conventionally measured by swinging the racquet about an axis in the handle located 10 cm (4 inches) from the end of the handle. In this paper, we will denote the conventional MOI of a bat or racquet as I_{15} or I_{10}, respectively, and following colloquial usage, will refer to it as the 'swing weight'. It can be measured by swinging the implement in a vertical plane as a physical pendulum and measuring its period of oscillation. Together with the mass and center of mass location, the period determines the MOI about the pendulum axis. Alternatively, the implement can be swung in a horizontal plane using a calibrated spring to provide a restoring force, in which case there is no need for any additional measurements of the mass and center of mass location to determine the swing weight.

There is an established tradition in golf that the numbered irons in a set of clubs differ in length by increments of 1.3 cm (0.5 inches) and that each club is matched by having the same first moment of the mass distribution, denoted herein by S_1, relative to an axis in the handle located 35.5 cm (14 inches) from the end of the handle [8]. The matching can be achieved by increasing the mass of the club head in increments of approximately 7 g (1/4 oz) to compensate for the decreasing length of each club. It was established in the 1930s that clubs matched in this manner feel much the same when they are

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swung and it has been that way ever since. However, a recent innovation by some club designers has been to match the MOI of a set of clubs, which can be achieved either by increasing the mass of the club head in increments of approximately 8 g, or by changing the lengths between clubs in increments of 0.95 cm (3/8 inches). The original choice of the first-moment fulcrum, 35.5 cm from the end of the handle, was not entirely arbitrary. It was a choice that happened to give an approximate MOI match and one that was easier to implement in the 1930s since it involved simple measures of mass and length rather than a time-consuming measurement of the actual MOI of all the clubs in a set. As a result, the MOI of a club has now become part of the lexicon among golfers and it is conventionally measured about an axis through the end of the handle. We will denote this quantity as I_0. We show in Section 4.3 that it is possible to match simultaneously both the S_1 and I_0 of a set of clubs simply by changing the fulcrum from 35.5 to 47.0 cm from the handle end.[1]

In this paper, we address the importance of the MOI and coefficient of restitution (COR) to the performance of a sporting instrument. We start with a discussion of intrinsic power and effective mass in Section 2 and its dependence on the MOI. In Section 3, we discuss the relationship between swing speed and MOI. We then apply these ideas to specific examples of tennis, baseball, and golf in Section 4. A summary is given in Section 5.

2. INTRINSIC POWER AND EFFECTIVE MASS OF AN IMPLEMENT

A powerful bat or racquet or club can be defined as one that projects a ball at high speed. A less powerful implement is one that projects the ball at a lower speed, given the same incoming ball speed. These factors can be quantified in terms of the simple impact model shown in Figure 1. Suppose that a ball of mass m is incident at speed v_1 and is struck by an implement of mass M swung at speed V_1. The implement speed V_1 is taken to be the speed at the point of impact, located at distance b from the center of mass. If the ball is struck head on, then it will exit back along the incident path at speed v_2, and the impact point on the bat will decrease in speed to V_2 at the end of the impact.

The collision process is complicated by the fact that the implement is not a rigid body on the short time scale of the collision. Energy can be transferred to the implement in the form of transverse bending vibrations. Nevertheless, the exit speed of the ball is governed by the same momentum conservation laws that would apply even if the implement were perfectly rigid. Therefore, a flexible implement can be described as though it were a rigid body, with a reduced value of the COR to allow for vibrational energy losses in the implement. If the impact occurs at the fundamental vibration node

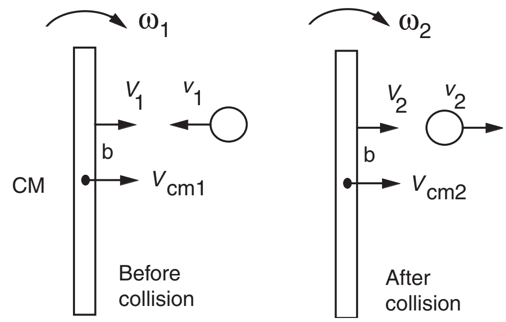

Figure 1. Ball incident at speed v_1 is struck head on by an implement swung at speed $V_1 = V_{cm1} + b\omega_1$. The ball exits at speed v_2. The exit speed depends on both V_1 and the 'intrinsic power' of the implement itself. CM denotes center of mass.

point, then the fundamental vibration mode is not excited at all, in which case the vibration energy losses in the implement are much reduced and the primary energy loss during the collision is the energy loss in the ball. A significant reduction in the outgoing ball speed can arise if the ball is struck at a point that does not coincide with the long axis of the implement, causing the implement, the hand and the forearm to rotate and vibrate about that axis. However, in this paper we assume that the ball impacts along the central axis of the implement.

Because the collision with the ball takes place over a very short time, the force F at the impact point is generally much larger than the force applied by the player at the handle end. In the following, we ignore the force acting at the handle end, which is equivalent to assuming that the bat is free on the short time scale of the collision. In that case, the implement experiences both a center of mass acceleration according to the relation $F = MdV_{cm}/dt$, and an angular acceleration according to the relation $Fb = I_{cm}\, d\omega/dt$, where I_{cm} is the MOI of the implement about an axis through its center of mass. The impact point itself accelerates according to the relation $F = M_e dV/dt$, where M_e is the effective mass of the impact point. Since $V = V_{cm} + b\omega$, we find that:

$$\frac{1}{M_e} = \frac{1}{M} + \frac{b^2}{I_{cm}}. \quad (1)$$

Therefore, the collision between the implement and the ball can formally be regarded as a head-on collision between a point mass M_e with incident speed V_1 and a point mass m at incident speed v_1, with a COR $e = (v_2 - V_2)/(v_1 + V_1)$. If the implement is initially at rest, then the ball will rebound with a speed ratio given by:

$$e_A = \frac{v_2}{v_1} = \frac{eM_e - m}{M_e + m}, \quad (2)$$

where e_A is known as the apparent COR, although it is also known as the collision efficiency [4] or the rebound power. [8] If the implement is swung at an incoming ball, then the outgoing ball speed is given by:

$$v_2 = (1 + e_A)V_1 + e_A v_1. \quad (3)$$

The outgoing speed has two components, the first being directly proportional to the swing speed of the implement and the second representing the bounce speed of the incoming ball off the implement. Since both components depend on the value

[1] It has become conventional in golf for swing weight to refer to the first moment S_1 rather than the second moment I_0 of the mass distribution. To avoid confusion, we will not use the term 'swing weight' when referring to a golf club.

of e_A, the performance of any given implement depends on e_A, and thus, on the effective mass of the implement and the COR at the impact point. The quantity e_A can be regarded as a measure of the intrinsic power (or the 'inbuilt power') of the implement since it determines the rebound speed of the ball when the implement is initially at rest. The performance of the implement also depends on the speed at which it can be swung, which will be discussed in Section 3.

After the collision, the combined effects of rotation about the center of mass and translation of the center of mass of the free implement during the impact will result in the rotation of the implement about an axis located near the handle end, at a distance $x = I_{cm}/Mb$ from the center of mass, or at a distance $R = x+b$ from the impact point. The impact point is known as the center of percussion for that axis, and the axis is known as the conjugate point. The MOI of the implement about the axis is given by $I_A = I_{cm} + Mx^2$. It is easy to show that I_A is also equal to M_eR^2. In terms of the rotation induced by the collision, the implement behaves in the same manner as an isolated mass M_e located at the end of massless rod of length R.

The effective mass of the implement at the impact point can therefore be expressed in the form $M_e = I_A/R^2$, indicating that the intrinsic power of the implement is directly related to its MOI about the collision-induced rotation axis near the handle end. Given that M_e and the location of the rotation axis both depend on three separate and independent inertial properties of the implement (M, b, and I_{cm}), it might appear that the performance of any given implement should depend on all three of these properties rather than any single parameter. However, the value of I_A/R^2 for any given implement is relatively insensitive to the axis chosen to measure I_A and R, since I_A is proportional to R^2 to a good approximation, at least for the usual situation where a ball impacts near one end of an implement and the implement rotates about an axis near the other end. It is for this reason that the intrinsic power of any given implement depends primarily on the conventional MOI of the implement about an axis near the handle end, regardless of the precise location of the collision-induced rotation axis.

We illustrate this point with some examples. Suppose that the implement is a uniform rod of length L and mass M with its center of mass in the middle of the rod and with $I_{cm} = ML^2/12$. Such an implement is a good approximation to a tennis racquet, as discussed more fully in Section 4.1. If the impact occurs at a distance d from one end, then $b = L/2-d$, and the effective mass at the impact point is given by:

$$M_e = \frac{M}{1 + 3(1 - 2d/L)^2}. \quad (4)$$

For an impact at the center of the rod, $M_e = M$, while $M_e = M/4$ for an impact at the tip of the rod. Now suppose that we choose to evaluate I_A about an arbitrary axis located at a distance A from the handle end of the rod, or a distance $L/2-A$ from the center of the rod, and a distance $R = L-A-d$ from the impact point. For this axis, $I_A = ML^2/12 + M(L/2 - A)^2$. The quantity I_A/R^2 can be taken as an approximate estimate of M_e and is compared with the actual value of M_e in Figure 2 for two different values of A. If the axis is chosen at $A/L = 0.1$, then the estimated value of M_e is within

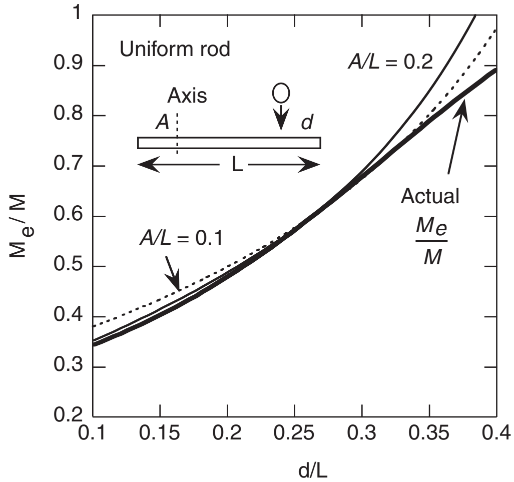

Figure 2. Estimated $M_e/M \approx I_A/MR^2$ and actual M_e/M (thick curve) *vs* d/L for a uniform rod of length L and mass M, when I_A is measured about an axis at distance $A/L = 0.1$ (dashed curve) or 0.2 (thin curve) from the handle end. d is the distance from the impact end to the impact point.

2% of the actual value of M_e for all impacts in the range $0.22 < d/L < 0.35$. If the axis is chosen at $A/L = 0.2$, then the estimated value of M_e is within 2% of the actual value of M_e for all impacts in the range $0.13 < d/L < 0.28$.

A similar result is obtained if the implement has a light handle and a heavy head, as is the case for a golf club or baseball bat, although a more realistic example for a bat will be given in Section 4.2. Consider a club of total mass M having a head of zero extent and mass M_h located at the end of a uniform shaft of length L and mass m. The center of mass of the club is located at a distance $b = m/L(2M)$ from the head, and $I_{cm} = 0.25mL^2/(1/3 + M_h/M)$. From Equation 1 it can be seen that the effective mass at the head end of the club is $M_e = M_h + m/4$, equal to the mass of the head plus the effective mass of the shaft at its far end. If we measure I_A about an axis located an arbitrary distance A from the handle end or a distance $R = L-A$ from the head end, then M_e is given approximately by I_A/R^2, regardless of the chosen axis, as shown in Figure 3. I_A/R^2 is exactly equal to M_e for an axis at the conjugate point (at a distance of $I_{cm}/(Mb)$ from the center of mass) and differs by less than 2% from M_e for any other axis located in the range $0.05 < A < 0.48$ m.

3. SWING SPEEDS

As we have seen from Equation 3, the performance of a sporting instrument, defined as the outgoing ball speed, depends on both the intrinsic power e_A of the instrument and the speed with which it is swung V_1. We have also seen that e_A depends on the conventional MOI I_A of the implement about an axis near the handle end. It is often argued that light instruments can be swung faster than heavy implements to make up for their lack of intrinsic power. In this section, we discuss briefly the results of several experiments showing that V_1 depends on the conventional MOI rather than the mass.

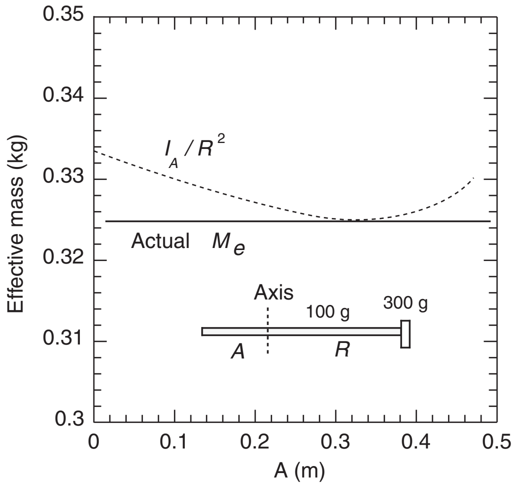

Figure 3. I_A/R^2 *vs* A for a club of length $L = 1$ m with a uniform shaft of mass 0.1 kg and a head of mass 0.3 kg. Effective mass of the club at the head end is 0.325 kg. I_A/R^2 is equal to M_e at $A = 0.333$ m and differs from M_e by less than 2% for any axis in the range $0.05 < A < 0.48$ m.

The most definitive evidence comes from an experiment by Smith *et al.* [10] using a series of 16 different softball bats swung by 20 different elite softball players. The bats were specially constructed for the study so as to distinguish a dependence of swing speed on mass from MOI. Therefore, 10 of the bats had the same mass and different values of the MOI, while the other 10 had the same MOI and different mass. The results showed that the maximum swing speed, V, for any given player depended on the MOI about an axis through the handle end of the bat, I_0, according to the relation $V \propto 1/I_0^n$, where $n = 0.25$, averaged over all players. Furthermore, swing speed was found to be independent of bat mass when I_0 was held fixed. In a similar experiment conducted by Cross and Bower [11] using a series of rods with a much wider range of I_0 values than used in the Smith *et al.* study, it was found that $n = 0.27 \pm 0.01$, averaged over four participants. The earliest experiment of this type was probably the one conducted by Daish and described in his 1972 book [5]. Daish measured the head speed of golf clubs as a function of head mass, finding that head speed decreases as head mass increases for any given player performing a maximum effort swing. When reinterpreted in terms of MOI, the Daish study [5] is consistent with the more recent studies, with $n = 0.26$, averaged over five golfers.

An extensive study of swing speeds using baseball bats was undertaken by Crisco and Greenwald [12]. They found that good batters swung bats in such a way that, just before impact with the ball, the bat swings in a circular arc centered within an inch or two of the knob. They also found that the angular velocity ω of a bat just before impact, averaged over several strong batters, is given by:

$$\omega = \frac{31.1\ \text{rad/s}}{(I_0)^{0.28}} \tag{5}$$

with I_0 in kg-m^2. A similar result was found by Fleisig *et al.* [13].

From a physics point of view, it is the second moment (or MOI) rather than the first moment that determines the resistance of an object to rotation. It is not immediately obvious that the swing speed of an implement should also depend primarily on its second moment, given that the person swinging the implement must swing his or her relatively heavy arms at the same time. Nevertheless, the measured swing speed results show that this is the case. Consequently, one might expect that the swing speed of a golf club should also depend primarily on its MOI rather than its first moment.

We now show how the first moment S_1 of a set of clubs can be matched in such a way that I_0 is also matched. Suppose that a club has a uniform shaft of mass m and length L, with a point mass M_h located at the far end. Then S_1 about a point at distance A from the handle end is given by:

$$S_1 = (0.5L - A)m + (L - A)M_h, \tag{6}$$

while I_0 is given by

$$I_0 = \left(\frac{m}{3} + M_h\right)L^2 = L^2\left[\frac{m}{3} + \frac{S_1 - (0.5L - A)m}{L - A}\right]. \tag{7}$$

I_0 does not dependent explicitly on m if $A = L/4$, in which case $I_0 = (4L/3) \times S_1$. Since a set of irons is typically about 91 cm in length, a measurement of S_1 with a 23 cm fulcrum would indeed provide a good estimate of I_0, regardless of the mass of the shaft or the head, at least for a uniform shaft. However, I_0 is proportional to the length of a club for any given S_1. Consequently, such a measurement is not well suited to matching the second moment of a set of clubs, due to the small variation in length between clubs.

If each club varies in length by an increment ΔL and if the head mass is altered by ΔM_h, then the resulting changes in S_1 and I_0 are given by:

$$\Delta(S_1) = (L - A)\Delta M_h + \left(M_h + m - \frac{Am}{L}\right)\Delta L \tag{8}$$

$$\Delta I_0 = L^2 \Delta M_h + (2M_h + m)L\Delta L. \tag{9}$$

For any particular value of A, it is possible to match a set of clubs so that $\Delta(S_1) = 0$, but ΔI_0 will not be zero in general. However, if $A = L/2$, then $\Delta I_0 = 0$ when $\Delta(S_1) = 0$. Consequently, the MOI of a set of clubs can be matched by matching the first moment about a fulcrum located halfway along the club. This method of matching has not previously been noticed. A practical example is described in Section 4.3, where it is shown that the matching technique remains valid even if the shaft is non-uniform, as it normally is.

4. APPLICATIONS

4.1 Tennis Racquets

A tennis racquet has an approximately uniform mass distribution along its central axis, and therefore, behaves in a manner similar to a uniform rod. The length of a tennis racquet is typically about 69 cm (27 inches). The conventional MOI (or swing weight) is measured about an axis 10 cm from the end of the handle, so that $A/L = 0.15$. Since most impacts occur in the range $0.1 < d/L < 0.3$, the swing weight of a racquet provides a convenient and reliable measure of its effective mass for almost all impacts of practical interest. Denoting the swing weight by

I_{10}, then the effective mass at a point near the middle of the strings, 16 cm from the tip, is given to a very good approximation by $M_e \approx I_{10}/(L-26)^2$ when L is measured in centimeters and I_{10} is measured in kg-cm^2. The estimated value of M_e here does not depend on the location of the center of mass of the racquet. It depends only on the swing weight and the distance, $L-26$, between the impact point and the axis chosen to measure the swing weight. The middle of the strings is close to the fundamental vibration node, in which case the energy loss due to frame vibrations is negligible, and the corresponding value of e_A can be estimated directly from Equation 3 using the known value of e for the impact of a tennis ball on the strings of a racquet (approximately 0.85) and the estimated value of M_e.

The application of these ideas to actual racquets is presented in Figure 4. Plotted are the calculated value of e_A for 133 different racquets, all available for sale during 2005 and all 69 ± 0.5 cm long, for an impact 16 cm from the tip of the racquet, as a function of (a) racquet mass and (b) the measured swing weight of each racquet. The measured distance between the end of the handle and the center of mass for all 133 racquets varied from 30.5 to 37.5 cm. It is clear from these results that the intrinsic power of a racquet at a point near the middle of the strings depends on the swing weight of the racquet and that the correlation with racquet mass is weak. Heavy racquets are designed with most of the additional weight at the handle end, while light racquets are designed by removing weight mostly from the handle end. Weight added or removed from the handle affects the overall weight of a racquet, but has little effect on the MOI or on the intrinsic power of the racquet. The MOI about an axis through the handle depends mainly on the mass of the head, as does the effective mass at the impact point, thus the strong correlation between racquet performance and the measured swing weight.

4.2 Baseball Bats

For baseball bats, the MOI (or swing weight) is conventionally measured about an axis on the handle of the bat a distance 15 cm (6 inches) from the knob end and is denoted by I_{15}. Published values of the swing weight for a large range of baseball and softball bats are not readily available. However, it is easy to calculate the swing weight for a large range of bats in terms of a simple bat model. For that purpose, a baseball bat of length 0.84 m (33 inches) was modeled as three equal length uniform cylindrical sections, each section having a different and progressively larger mass and diameter from the handle end to the barrel end. The mass of each section was varied in 28 g (1 oz) increments to model a total of 556 different bats varying in overall mass from 0.60 to 0.96 kg (21–34 oz). For each bat, the quantities M_e and e_A were calculated at an impact point 15 cm (6 inches) from the end of the barrel. The fundamental node point is typically at this location, so that vibration losses in the bat can be ignored. Therefore, the value of the COR e is determined primarily by losses in the ball, and varies from approximately 0.47 to approximately 0.50 for a high-speed collision with a wood bat, but can be significantly larger for aluminum bats due to the trampoline effect [4]. In our model calculations, two values of e were chosen, 0.50 and 0.55, the former and latter appropriate for a wood and aluminum bat, respectively.

The value of M_e for all 556 bats is shown in Figure 5a as a function of the barrel mass, demonstrating that the effective mass of the bat at the impact point depends primarily on the mass of the barrel section rather than the middle or handle sections of the bat. In Figure 5b, an approximate effective mass M_e^* is plotted versus the exact value M_e, where $M_e^* = I_{15}/R^2$ and R is the distance between the impact point and a point on the handle 15 cm from the knob ($R = 0.69$ m in our example). As with the examples in Figures 2 and 3, M_e^* is an excellent approximation to M_e over the full range of bats in the model. As a result, we expect a strong correlation between the intrinsic power e_A and the swing weight I_{15}, as clearly demonstrated in Figure 5d. We expect and see no such correlation with the total bat mass M (Figure 5c). The value of e_A for each bat is shown in Figure 5d for two different values of e, showing how the trampoline effect enhances the performance of aluminum bats. By comparison, Figure 6 shows e_A for 112

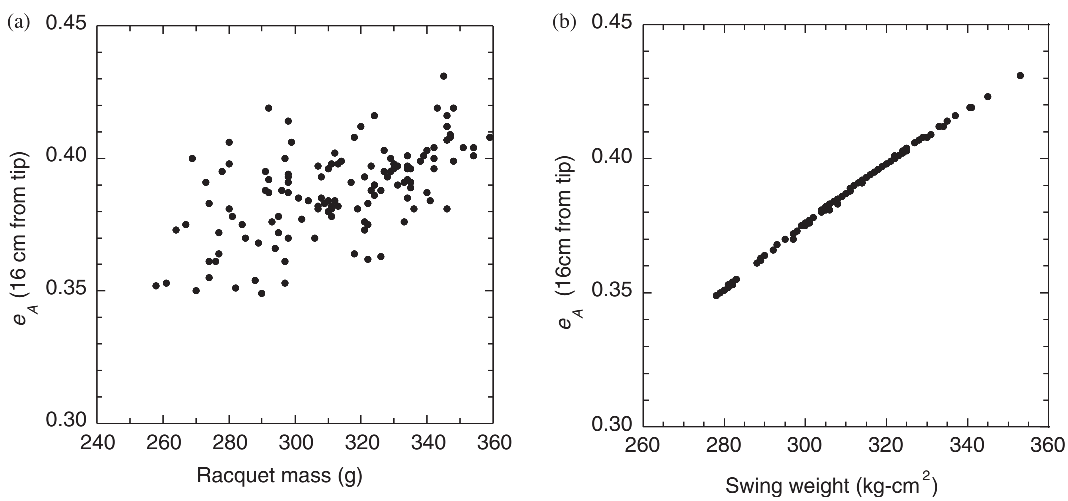

Figure 4. e_A at a point 16 cm from the tip of a racquet *vs* (a) racquet mass M and (b) swing weight I_{10} of each racquet, when $e = 0.85$ for 133 different racquets all 27 inches long. Data show the strong correlation of e_A with swing weight and weak correlation with mass.

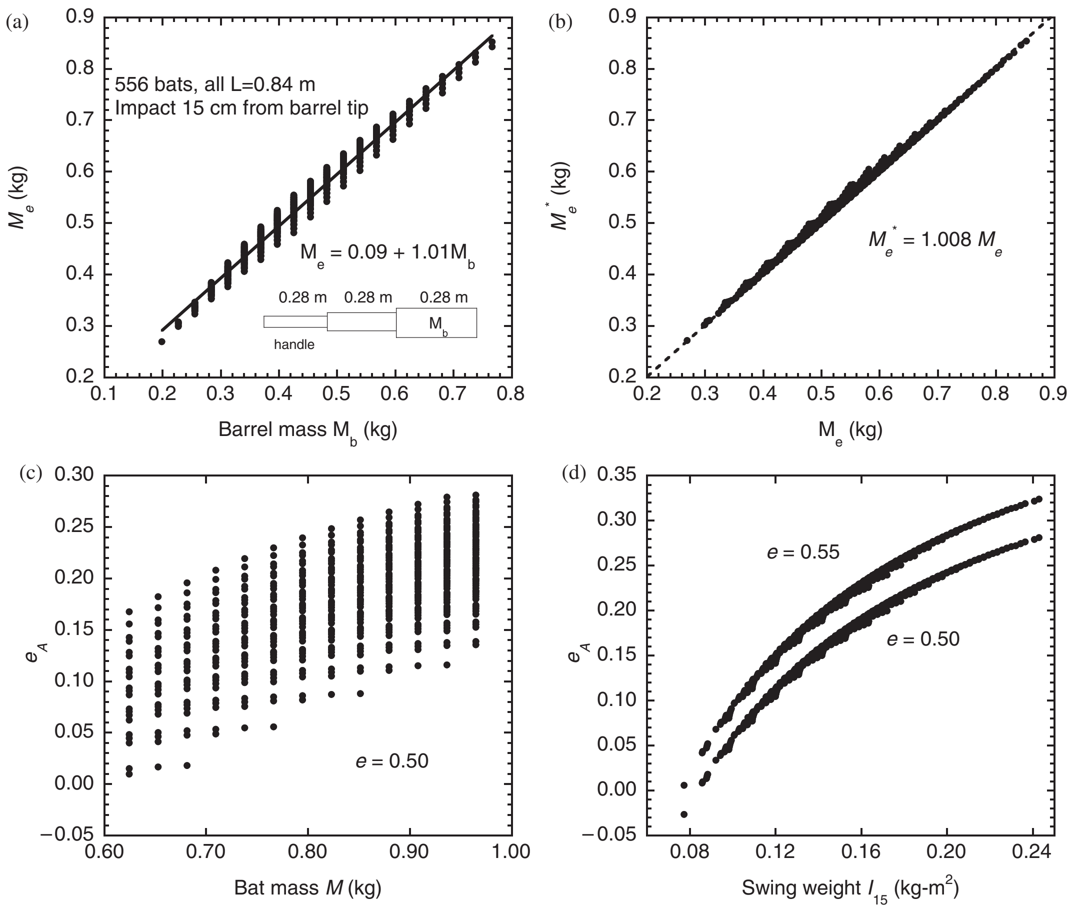

Figure 5. (a) M_e at a point 15 cm from the tip of the barrel *vs* barrel mass M_b; (b) approximate effective mass $M_e^* = I_{15}/(0.69\,\mathrm{m})^2$ *vs* M_e; (c) e_A at a point 15 cm from the tip of the barrel *vs* bat mass M; (d) e_A *vs* swing weight I_{15} for 556 different bats, all 0.84 m long.

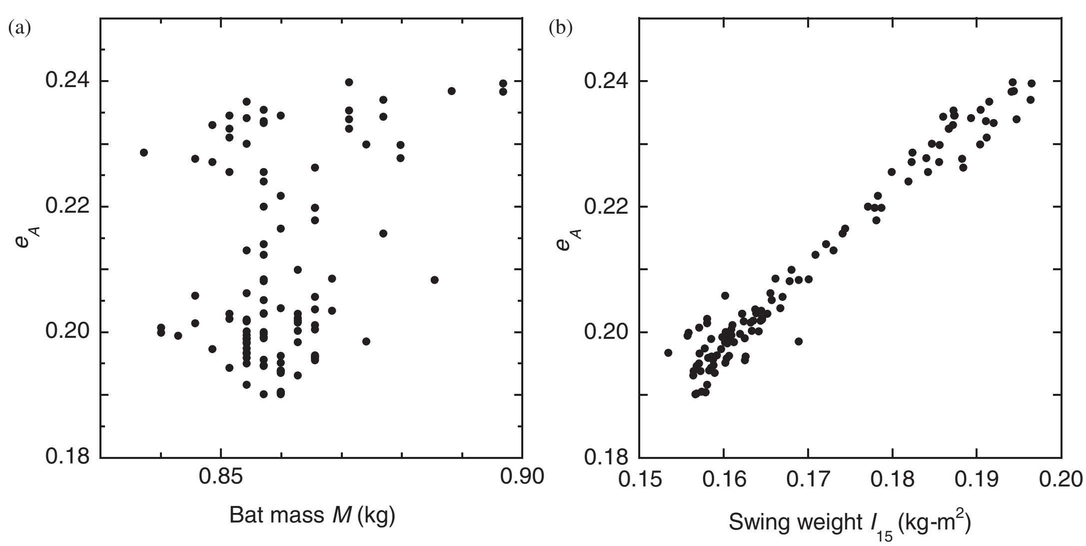

Figure 6. e_A at a point 15 cm from the tip of the barrel *vs* (a) bat mass M and (b) swing weight I_{15} for 112 different bats all 0.84 m long, that are approved for play in National Collegiate Athletic Association games. Coefficient of restitution was fixed at 0.50.

actual bats that are approved for play in the USA by the National Collegiate Athletic Association (NCAA). All the bats have a length in the range 0.84 ± 0.01 m, a weight in the range 0.84–0.88 kg (30–31 oz), and a swing weight between 0.15 and 0.20 kg-m^2 (8500–11000 oz-in^2). We reach the same conclusion for these actual bats that we did for our model bats: e_A is strongly correlated with swing weight, but poorly correlated with the actual weight.

As discussed earlier in the context of Equation 3, the performance of a bat (that is, batted ball speed), depends not only on the intrinsic power e_A, but also on the speed with which the bat can be swung. We have seen that e_A depends on the swing weight. Moreover, experimental data, as summarized in Equation 5, also indicate that the swing speed depends on the swing weight. We therefore expect a dependence of bat performance on the swing weight, a topic that we now address. Guided by Figure 5b, we use M_e^* to calculate e_A. To calculate the swing speed, we use Equation 5 along with a simple empirical expression relating I_0 to I_{15} for our model bats. We assume a fixed value of 0.50 for the COR and three different values of the incoming ball speed. The results of this exercise are presented in Figure 7, where we show how the batted ball speed depends on I_{15}. A similar plot is shown in Figure 8 for the 0.84 m NCAA bats, where the exact value of M_e and I_0 were used, along with two different values for the COR.

The results in Figure 7 show that the batted ball speed increases rapidly, levels off, then falls slowly as the swing weight I_{15} is increased. This behavior is easy to understand as a tradeoff between e_A and swing speed, which depend on the swing weight in opposite ways. A bat with a very small swing weight, such as a broomstick, would be easy to swing, but would have a small intrinsic power. A bat with a very large swing weight, such as a 10 kg steel bar, would be much more difficult to swing, but would have a larger intrinsic power. The optimum swing weight, that is, the one producing the largest batted ball speed, would lie somewhere between the two extremes and depends somewhat on the incoming ball speed.

Interestingly, the data in Figure 8 show that actual bats used in amateur play tend to have a swing weight a bit smaller than the optimum, on the rising part of the curve in Figure 7, suggesting that batters could improve their maximum batted ball speed by using a bat with a larger swing weight. Judging from their selection of bats, they prefer not to do so. Batters distinguish between bat speed and bat quickness. The former has to do with the bat speed at the moment of the collision. The latter has to do with the bat acceleration, which affects the batter's ability to control the movement of the bat and get it into the hitting zone quickly. So while a batter can hit a ball harder with a swing weight near the top of the curve, he is likely to hit a ball solidly more often with a somewhat smaller swing weight. The strong preference by batters to use a less than optimum swing weight provides a logical explanation for the NCAA rule that specifies a lower limit but not an upper limit on the allowable swing weight of a bat.

It is also interesting to point out that bat control is generally not an issue for slow-pitch softball, given the very low speed of the incoming ball. Indeed, there is anecdotal evidence that elite slow-pitch softball players prefer bats with a larger swing weight, a preference consistent with our findings. A related issue has to do with the illegal act of corking a baseball bat, whereby the swing weight of a wood bat is reduced by drilling a ~2.5 cm diameter hole axially in the barrel of the bat to a depth of about 25 cm, then backfilling the hole with a light material, such as cork. It is often claimed by baseball aficionados that the batter using a corked bat will hit the ball harder because the increase in swing speed more than compensates for the decrease in intrinsic power e_A, leading to a larger batted ball speed. Given that bats in use fall on the rising part of the curve in Figure 7, our analysis leads to the opposite conclusion: corking the bat leads to a lower maximum batted ball speed. Nevertheless, corking a bat is not without value, since the reduced swing weight allows better bat control.

The calculations in Figure 8 indicate that batted ball speed is a much stronger function of the COR than of the swing weight. Indeed, the performance of bats at the top of the curve in Figure 7 would be independent of swing weight, at least over some limited range. An inspection of Figure 8 indicates that a 10% change in e leads to a change in batted ball speed of 2.7 m/s, whereas a 10% change in swing weight leads to a change of only 0.7 m/s. The weak dependence of batted ball

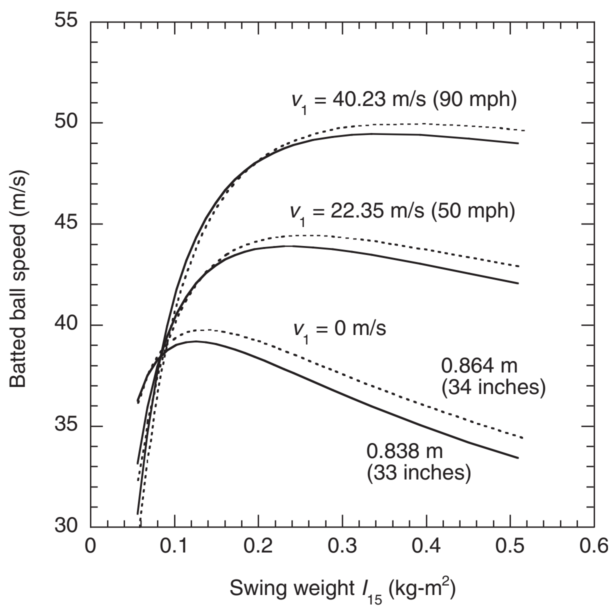

Figure 7. Batted ball speed v_2 *vs* swing weight I_6 for 0.84 and 0.86 m baseball bats and for three different values v_1 of the incoming ball speed.

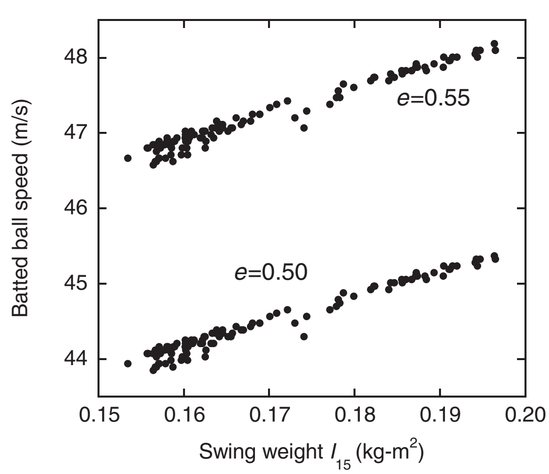

Figure 8. Calculated batted ball speed v_2 *vs* the swing weight I_{15} for 0.84 m bats approved for use in National Collegiate Athletic Association games. Two values of the coefficient of restitution are typical of wood and aluminum bats, respectively.

speed on swing weight and strong dependence on e has recently led the NCAA to adopt e as its metric of performance [14]. A different situation exists with tennis racquets since the strings provide a strong trampoline effect regardless of string tension. It was shown by Brody *et al.* [7] that a decrease in string tension from 267 to 223 N leads to an increase in serve speed of less than 1%. On the other hand, the change from a wood bat to a hollow aluminum bat leads to a much larger change in e, resulting in a much larger change in batted ball speed.

We conclude from our study that for a given pitch speed, bat length, and impact location, baseball bat performance is determined by only two properties of the bat: the swing weight I_{15} and the COR e. In particular, the performance does not directly depend on the total mass of the bat or the location of either the center of mass or the center of percussion. From a purely technological point of view, this conclusion simplifies the job of the bat designer, since the swing weight and COR are independently-adjustable parameters. The COR is largely determined by the stiffness of the barrel, which determines the size of the trampoline effect. The swing weight can be adjusted by adding or removing weights from the barrel endcap. For example, adding or removing 28 g (1 oz) at the endcap of a 0.84 m bat changes the swing weight by ± 0.0133 kg-m^2 without affecting appreciably the COR.

4.3 Golf Clubs

In this section, we consider the problem of matching the numbered irons in a set of clubs so that they all have the same MOI I_0. A typical set of irons consists of a number 3 iron of length approximately 99 cm, decreasing in length by 1.3 cm increments to a pitching wedge of length approximately 90 cm. In a conventional set with matched first moments, the mass of the head increases in 7 g increments from approximately 240 g to approximately 289 g, although lighter and heavier ranges of head weights are available to suit different players. If the shaft were a uniform cylinder of mass approximately 110 g, then a first moment-matched set would require an incremental head mass of approximately 5 g rather than 7 g. However, the handle end of the shaft is normally thicker and heavier than the head end and includes a grip of length approximately 25 cm and mass approximately 50 g. Table 1 shows the calculated parameters of a matched set of clubs with a steel shaft consisting of a uniform cylindrical handle of length 30 cm and mass 100 g, plus a uniform cylindrical section of mass 1.15 g/cm joining the handle and the head.

Three different matching methods are shown in Table 1, all with the same 240 g head mass for the number 3 iron. Only the number 3 iron, the number 6 iron, and the shortest iron (the pitching wedge) are shown in the Table. The first method is the conventional one where the first moment is matched about a fulcrum located 35.5 cm from the end of the handle. The first moment is 0.155 kg-m for each iron, while the MOI about an axis at the end of the handle, I_0, decreases by 0.009 kg-m^2 from the first to the last iron. The second system is an MOI matched set where I_0 is the same for each club, but the first moment increases by 0.006 kg-m from the first to the last iron. The third matching system is one where the first moment is matched about a fulcrum 47 cm from the end of the handle. In the third case, I_0 is matched closely for all clubs, while the first moment is matched exactly. In the third system, I_0 varies by only 0.15% at most between the seven clubs in the set, the average increment in head mass being 8.7 g. The third system is almost identical to the second in terms of the resulting head weights, the only significant difference being that the first moment is defined with a different fulcrum, and therefore, has a different numerical value.

Table 1. Three matching methods for golf clubs. Units are kg-m for S_1, kg-m^2 for I_0, and cm for L and A.

Method	Iron	L	Head Mass	A	S_1	I_0
1	3	99.1	240	35.6	0.155	0.275
1	6	92.3	260	35.6	0.155	0.271
1	PW	90.2	290	35.6	0.155	0.266
2	3	39.0	240	35.6	0.155	0.275
2	6	92.3	264	35.6	0.157	0.275
2	PW	90.2	301	35.6	0.161	0.275
3	3	99.1	240	47.0	0.107	0.275
3	6	92.3	264	47.0	0.107	0.274
3	PW	90.2	301	47.0	0.107	0.275
3	3	99.1	200	47.0	0.092	0.235
3	6	92.3	219	47.0	0.092	0.234
3	PW	90.2	250	47.0	0.092	0.235

PW, pitching wedge.

The main point of these results is not to show that one matching system is better than another, but to show that it is possible to match both the first and second moment of a set of clubs, simply by choosing an appropriate axis to define each moment. By redefining the first moment fulcrum in this way, it is possible to match the MOI of a set of clubs to within 0.15% using a simple measurement of club mass and center of mass location, rather than measuring the actual MOI of each club in the set. The method also works for lighter or heavier sets of clubs using the same shafts, as shown in the last set of entries in Table 1. At present, MOI matching can be achieved only by employing a relatively expensive instrument designed specifically to measure the MOI of a club as it swings against a calibrated spring. Consequently, MOI matching is not as popular as perhaps it should be.

5. SUMMARY

In this paper, we have investigated the importance of the MOI of a sporting implement for its performance. We have shown that the intrinsic power e_A of a tennis racquet, baseball bat, or golf club is strongly correlated with the MOI of the instrument about an axis passing through the handle and poorly correlated with the total mass. For a tennis racquet, both e_A and the outgoing ball speed depend primarily on I_{10}, the MOI about an axis 10 cm from the handle end depends only weakly on the string tension. For a baseball bat, e_A depends primarily on I_{15}, the MOI about an axis on the bat 15 cm from the knob. For non-wood bats approved for use by the NCAA, the batted ball speed is a stronger function of the COR than of the MOI. For solid wood bats, it is the MOI that determines the performance since there is no significant trampoline effect with wood bats. We have also shown that it is possible to match simultaneously the MOI about the end of the handle I_0 and the first moment of a set of clubs by moving the fulcrum of the first moment from the conventional 35.5–47.0 cm from the handle. This matching method would provide players and golf technicians with a simpler and cheaper method of MOI matching, which is claimed by many players to result in significantly improved results on the golf course.

REFERENCES

1. Van Zandt LL. The dynamical theory of the baseball bat. *Americal Journal of Physics* 1992; **60**: 172–181.
2. Cross R. Impact of a ball with a bat or racket. *Americal Journal of Physics* 1999; **67**: 692–702.
3. Nathan A. Dynamics of the baseball bat collision. *Americal Journal of Physics* 2000; **68**: 979–990.
4. Nathan A. Characterizing the performance of baseball bats. *Americal Journal of Physics* 2003; **71**: 134–143.
5. Daish CB. *The physics of ball games*. English Universities Press: London, 1972.
6. Jorgensen T. *The physics of golf*, 2nd Ed. Springer–Verlag: New York, 1999.
7. Brody H, Cross R, Lindsey C. *The physics and technology of tennis*. Racquet Tech Publishing, Solana Beach, CA. 2002.
8. Cross R, Lindsey C. *Technical tennis*. Racquet Tech Publishing: Solana Beach, CA. 2005.
9. Adair R. *The physics of baseball*, 3rd Ed. Perennial: New York, 2002.
10. Smith L, Broker J, Nathan A. A study of softball player swing speed. In: Subic A, Trivailo P, Alam F, eds. *Sports Dynamics: Discovery and Application*. RMIT University: Melbourne, 2003; 12–17.
11. Cross R, Bower R. Effects of swing–weight on swing speed and racket power. *Journal of Sports Sciences* 2006; **24**: 23–30.
12. Greenwald RM, Penna LH, Crisco JJ. Differences in batted-ball speed with wood and aluminum baseball bats: a batting cage study. *Journal of Applied Biomechanics* 2001; **17**: 241–252.
13. Fleisig GS, Zheng N, Stodden DF, Andrews JR. Relationship between bat mass properties and bat velocity. *Sports Engineering* 2000; **5**: 1–14.
14. NCAA. Bat-testing regulations modified. National Collegiate Athletics Association, 2008. http://www.ncaa.org/wps/ncaa?ContentID = 38459 [March 2009].

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