I. INTRODUCTION

Particle scattering experiments have been conducted for many years to probe the structure of the atom, the atomic nucleus, and nucleons. In comparison, very few scattering experiments have been conducted with macroscopic objects. In this paper we describe an experiment on the scattering of a baseball by a baseball bat to determine how the speed and spin of the outgoing ball depends on the scattering angle. In principle, the results could be used to determine an appropriate force law for the interaction, but we focus attention on directly observable parameters. The main purpose of the experiment was to determine the amount of backspin that can be imparted to a baseball by striking it at a point below the center of the ball. The results are preliminary in that they were obtained at lower ball speeds than those encountered in the field. As such, the experiment can easily be demonstrated in the classroom or repeated in an undergraduate laboratory.

A golf ball is normally lofted with backspin so that the aerodynamic lift force will carry the ball as far as possible. For the same reason, a baseball will also travel farther if it is struck with backspin. It also travels farther if it is launched at a higher speed. In general there is a tradeoff between the spin and speed that can be imparted to a ball, which is affected in baseball by the spin and speed of the pitched ball. Sawicki, Hubbard, and Stronge concluded that a curveball can be batted further than a fastball despite the higher incoming and outgoing speed of the fastball. The explanation is that a curveball is incident with topspin and hence the ball is already spinning in the correct direction to exit with backspin. A fastball is incident with backspin so the spin direction needs to be reversed to exit with backspin. As a result, the magnitude of the backspin imparted to a curveball is larger than that imparted to a fastball for a given bat speed and impact point on the bat, even allowing for the lower incident speed of a curveball. The larger backspin on a hit curveball more than compensates for the smaller hit ball speed and a curveball travels farther than a fastball, a conclusion that has been extrapolated to the higher speeds that are typical of the game of baseball, they suggest that a curveball can be hit with greater backspin than a fastball, but by an amount that is less than would be the case in the absence of tangential compliance.

II. EXPERIMENTAL PROCEDURES

A baseball was dropped vertically onto a stationary, hand-held baseball bat to determine the rebound speed and spin as functions of the scattering angle and the magnitude and direction of spin of the incident ball. The impact distance from the longitudinal axis of the bat was varied randomly in order to observe scattering at angles up to about 120° away from the vertical. Measurements were made by filming each bounce with a video camera operating at 100 frames/s, although satisfactory results were also obtained at 25 frames/s. The bat was a modified Louisville Slugger model R161 wooden bat of length 84 cm (33 in.) with a barrel diameter of 6.67 cm (2.6 in.) and mass \( M = 0.989 \) kg (35 oz). The center of mass of the bat was located 26.5 cm from the barrel end of the bat. The moments of inertia about axes through the center of mass and perpendicular and parallel, respectively, to the longitudinal axis of the bat were 0.0460 and \( 4.39 \times 10^{-4} \) kg m². The ball was a Wilson A1010, with a mass 0.145 kg and diameter 7.2 cm.

The bat was held in a horizontal position by one hand and the ball was dropped from a height of about 0.8 m using the other hand. A plumb bob was used to establish a true vertical in the video image and to help align both the bat and the ball. The ball was dropped with or without spin. To spin the ball, a strip of felt was wrapped around a circumference and the ball was allowed to fall vertically while holding the top end of the felt strip. A line drawn around a circumference was used to determine the ball orientation in each frame in order to measure its spin. The impact distance along the axis was determined by eye against marks on the barrel to within \( =5 \) mm. If the ball landed 140–160 mm from the barrel end of the bat, the bounce was accepted. Bounces outside this range were not analyzed.

The velocity of the ball immediately before and after impact was determined to within 2% by extrapolating data from
at least three video frames before and after each impact. The horizontal velocity was obtained from linear fits to the horizontal coordinates of the ball and the vertical velocity was obtained from quadratic fits assuming a vertical acceleration of 9.8 m/s². Additional measurements were made by bouncing the ball on a hard wood floor to determine the normal and tangential coefficients of restitution, the latter defined in Eq. (1), and a lower limit on the coefficient of sliding friction \( \mu_s \) between the ball and the floor. The coefficient of restitution values were determined by dropping the ball with and without spin from a height of about 1.5 m to impact the floor at a speed of 5.6±0.3 m/s. The incident ball spin was either without spin from a height of about 1.5 m to impact the floor and tangential coefficients of restitution, the latter defined in Eq. (1), and a lower limit on the coefficient of sliding friction \( \mu_s \) between the ball and the floor. The coefficient of restitution values were determined by dropping the ball with and without spin from a height of about 1.5 m to impact the floor at a speed of 5.6±0.3 m/s. The incident ball spin was either without spin from a height of about 1.5 m to impact the floor

\[
e^x = \frac{v_{x2} - r\omega_2 - (V_x - R\Omega)}{v_{x1} - r\omega_1},
\]

where \( \Omega \) is the angular velocity of the ball about the longitudinal axis immediately after the collision; and the parameter \( D \). The two coefficients of restitution are defined in terms of the normal and tangential velocities of the impact point on the ball, relative to the bat, immediately after and immediately before the bounce.

The bounce can also be characterized in terms of apparent coefficients of restitution, ignoring recoil and rotation of the bat. That is, we can define the apparent normal COR \( e_A \) and the apparent tangential COR, \( e_T \), given by

\[
e_T = \frac{v_{x2} - r\omega_2}{v_{x1} - r\omega_1}.
\]

There are three advantages of defining apparent COR values in this manner. The first is that apparent COR values are easier to measure because there is no need to measure the bat speed and angular velocity before or after the collision (provided the bat speed is zero before the collision). The second advantage is that the batted ball speed can be calculated from the measured apparent COR values for any given initial bat speed simply by a change of reference frame. We discuss this calculation in Appendix B. The third advantage is that the algebraic solutions of the collision equations are considerably simplified and therefore more easily interpreted. Apparent and actual values of the COR are related by

\[
e_A = \frac{e_T - r_N}{1 + r_N},
\]

and

\[
e_T = \frac{e_T - r_T}{1 + r_T} + \frac{SD}{2r_T} \left( \frac{r_T}{1 + r_T} \right) v_{x1}(1 + e_A) v_{y1} - r\omega_1,
\]

where the recoil factors, \( r_T \) and \( r_N \), are the ratios of effective ball to bat masses for normal and tangential collisions, respectively. An expression for \( r_T \) was derived by Cross:

\[
e^x = \frac{e_T - r_N}{1 + r_N},
\]
and an expression for $r_x$ is derived in Appendix A:

$$r_x = m \left( \frac{1}{M} + \frac{b^2}{I_0} \right).$$  \hspace{1cm} (5)

In Eqs. (5) and (6) $m$ is the ball mass, $I_0$ and $I$, are the moments of inertia about an axis through the center of mass and perpendicular and parallel, respectively, to the longitudinal axis of the bat, and $b$ is the distance parallel to the longitudinal axis between the impact point and the center of mass. For the bat used in the experiments at an impact distance 15 cm from the barrel end of the bat, $r_x=0.188$ and $r_z=0.159$, assuming the bat is free at both ends. The exit parameters of the ball are independent of whether the handle end is free or hand-held, as described previously in Ref. 9 or 10. Equation (4) will not be used except for some comments in Sec. IV C and for comparison with Ref. 1 in which it was assumed that $e_x=0$ and $D=0$, implying $e_T=-0.14$ for our bat. As discussed more fully in Sec. IV A, we find better agreement with our data with $e_T=0$.

From the definition of the parameter $D$, the normal force exerts a torque resulting in a change in angular momentum of the ball about the contact point given by

$$\left(I \omega_2 + mrv_{z_2}\right) - \left(I \omega_1 + mrv_{z_1}\right) = -D \int N dt = -mD(1 + e_A)v_{y_1},$$

where $I=amr^2$ is the moment of inertia of the ball about its center of mass. For a solid sphere, $\alpha=2/5$, although Brody has recently shown that $\alpha=0.378$ for a baseball.\textsuperscript{11} Equations (2) and (7) can be solved to show that

$$\frac{v_{z_2}}{v_{z_1}} = \frac{1 - \alpha e_T}{1 + \alpha} + \frac{\alpha(1 + e_T)}{1 + \alpha} \left( \frac{r \omega_1}{v_{z_1}} \right) - \frac{D(1 + e_A)}{r(1 + \alpha)} \left( \frac{v_{z_1}}{v_{z_1}} \right),$$  \hspace{1cm} (8)

and

$$\frac{\omega_2}{\omega_1} = \frac{\alpha - e_T}{1 + \alpha} + \frac{(1 + e_T)}{(1 + \alpha)} \left( \frac{v_{z_1}}{\omega_1} \right) - \frac{D(1 + e_A)}{r(1 + \alpha)} \left( \frac{v_{z_1}}{\omega_1} \right).$$  \hspace{1cm} (9)

Equations (8) and (9), together with the definition of $e_A$, give a complete description of the scattering process. For given initial conditions there are three observables, $v_{z_2}$, $v_{z_1}$, and $\omega_2$, and three unknown parameters, $e_A$, $e_T$, and $D$, that can be inferred from a measurement of the observables.

We have written Eqs. (8) and (9) for the general case of $D>0$. However, as we will show in Sec. IV A, the present data are consistent with $D=0$, implying conservation of the ball’s angular momentum about the point of contact. The normal bounce speed of the ball is determined by $e_x$, and for $D=0$ the spin and tangential bounce speed are determined by $e_T$ and $r\omega_1/v_{z_1}$. Depending on the magnitude and sign of the latter, $v_{z_2}$ and $\omega_2$ can each be positive, zero, or negative. Equations (8) and (9) are generalizations of equations derived by Cross\textsuperscript{4} for the special case of the ball impacting a massive surface and reduce to these equations when $e_A=e_x$ and $e_T=e_x$.

IV. EXPERIMENTAL RESULTS AND DISCUSSION

A. Determination of $e_T$

We initially analyze the data using Eqs. (8) and (9) assuming $D=0$ and postpone a discussion of angular momentum conservation. Results obtained when the ball is incident on the bat without initial spin are shown in Fig. 2. The ball impacted the bat at speeds varying from 3.8 to 4.2 m/s, but the results in Fig. 2 were scaled to an incident speed of 4.0 m/s by assuming that the rebound speed and spin are both linearly proportional to the incident ball speed, as expected theoretically. An experimental value $e_A = 0.375\pm0.005$ was determined from results at low (back) scattering angles, and this value was used together with Eqs. (8) and (9) to calculate the rebound speed, spin, and scatter-
ing angle as functions of the impact parameter for various assumed values of $e_T$. The best fits to the experimental data were found when $e_T=0$, but reasonable fits could also be obtained with $e_T=0\pm0.1$.

Results obtained when the ball was incident with topspin or backspin are shown in Fig. 3. These results are not expected to scale with either the incident speed or incident spin and have not been normalized. Consequently the data show slightly more scatter than those presented in Fig. 2. The ball impacted the bat at speeds varying from 3.9 to 4.1 m/s and with topspin varying from 79 to 83 rad/s or with backspin varying from −72 to −78 rad/s. Simultaneous fits to all three data sets resulted in $e_A=0.37\pm0.02$ and $e_T=0\pm0.02$. Using the recoil factors $r_x=0.188$ and $r_z=0.159$, our values for $e_A$ and $e_T$ imply $e_x=0.63\pm0.01$ and $e_z=0.16\pm0.02$. The result for $e_x$ is consistent with that measured by impacting the ball onto a hard floor ($0.17\pm0.03$), but the result for $e_z$ is slightly higher, presumably because of the lower impact speed and the softer impact on the bat. On the other hand, Figs. 2 and 3 show that the measured $\omega_2$ values are inconsistent with $e_T=-0.14$, which is the result expected for the bat if $e_z=0$, as assumed in Ref. 1.

We next investigate the more general case in which angular momentum is not conserved by fitting the data to Eqs. (8) and (9) allowing both $D$ and $e_T$ as adjustable parameters. By fitting to all three data sets simultaneously, we find $e_T=0\pm0.02$ and $D=0.21\pm0.29$ mm, thereby justifying our earlier neglect of $D$ and confirming that the data are consistent with angular momentum conservation. All the following calculations assume $D=0$.

It is possible to determine the incident and outgoing angles with respect to the normal, $\theta_1$ and $\theta_2$, from the measured quantities $v_1$, $v_2$, $\omega_1$, $\omega_2$, and $\alpha$ by applying angular momentum conservation about the contact point, Eq. (7), with $D=0$. Once $\theta_1$ and $\theta_2$ are known, it is possible to calculate the initial and final tangential velocities, which are plotted in Fig. 4. We see that the final tangential velocities are clustered around zero, as would be expected for $e_T=0$ [see Eq. (2)]. When plotted in this manner, it is clear that the data are inconsistent with $e_T=-0.14$. Note that the principal sensitivity to $e_T$ comes from large values of $|v_{x_1}-r\omega_1|$, which occurs whenever $v_{x_1}$ and $\omega_1$ have opposite signs and which leads both to a reversal of the spin and to scattering angles which are negative for $\omega_1>0$ and positive for $\omega_1<0$. Figure 3 shows that these regions have the greatest sensitivity to $e_T$.

**B. Implications for batted balls**

We next explore the implications of our results for the spin and speed of a batted ball, mindful that the present experiment was done at very low speeds compared to those appropriate for the game of baseball. The goal of this analysis is...
not to make any definitive predictions about the spin and speed of a batted ball but to examine the consequences of a small positive value of $e_T$ compared to the value $e_T=0$ assumed in Ref. 1. Whether such a value of $e_T$ is realized in a realistic ball-bat collision will have to await experiments at higher speed.

With this caveat, we first consider our data in Fig. 5, where we plot $\omega_2$ versus the impact parameter $E$, which is calculated from the inferred value of $\theta_i$. These data demonstrate that for a given $E > 0$, a ball with initial topspin ($\omega_1 > 0$) has a larger outgoing backspin than a ball with initial backspin, in qualitative agreement with Ref. 1. To investigate the argument more quantitatively, we compare the final spin on a fastball to that on a curveball. In this case the bat and ball approach each other prior to the collision, thereby requiring a change of reference frame to the equations we have derived. The relevant relations, Eqs. (B2), (B3a), and (B3b), are derived in Appendix B. We assume that the initial velocity of the bat is parallel to that of the ball, but displaced by the impact parameter $E$ as shown in Fig. 6. The initial bat speed is 32 m/s (71.6 mph). The incident fastball has a speed of 42 m/s (94 mph) and spin of $-200$ rad/s ($-1910$ rpm), and the incident curveball has a speed of 35 m/s (78 mph) and a spin of $+200$ rad/s. We use values of the normal COR $e_T$ assumed in Refs. 1 and 2 and present the calculated final spin as a function of $E$ in Fig. 6 for $e_T=0$, as determined from our measurements, and for $e_T=-0.14$, as assumed in Ref. 1.

Several important features emerge from this plot. First, the final spin $\omega_2$ is less sensitive to the initial spin $\omega_1$ for $e_T=0$ than for $e_T=-0.14$. This result is consistent with Eq. (9), where the first term on the right-hand side is larger for $e_T < 0$ than for $e_T=0$. Our result means that the difference in backspin between a hit fastball and hit curveball is not as large as has been suggested. Second, the gap between the spin on the curveball and fastball decreases as $E$ increases, a feature that can be understood from the second term on the right-hand side of Eq. (9). Because the initial speed is larger for the fastball than the curveball, the second term grows more rapidly for the fastball as $E$ increases, and the two curves cross at $E=2.4$ in. The rate at which the two curves converge is greater for $e_T=0$ than for $e_T=-0.14$. Had the initial speeds been identical, the two curves would have been parallel. Third, for $E \geq 0.5$ in. and independent of the sign of $e_T$, $\omega_2$ is larger when $e_T=0$ than when $e_T<0$, because $\omega_2$ is mainly governed by the second term on the right-hand side of Eq. (9). The increase in $\omega_2$ is accompanied by a decrease in $v_x$ as required by angular momentum conservation, and therefore by a slightly smaller scattering angle. The outgoing speed is dominated by the normal component, so the decrease in $v_x$ hardly affects the speed of the ball leaving the bat, at least for balls hit on a home run trajectory.

These results have implications for whether an optimally hit curveball will travel farther than an optimally hit fastball. To investigate this issue in detail requires a calculation of the trajectory of a hit baseball, much as was done in Ref. 1. This calculation requires knowledge of the lift and drag forces on a spinning baseball. Given the current controversy about these forces, further speculation on this issue is beyond the scope of the present work.

It is interesting to speculate on the relative effectiveness of different bats regarding their ability to put backspin on a baseball. As we have emphasized, the effectiveness is determined by a single parameter, $e_T$, which is related to $e_T$ and the recoil factor $r_x$. For a given $e_T$, a bat with a smaller $r_x$ will be more effective than one with a larger $r_x$ [see Eq. (4)]. Because $r_x$ is dominated by the term involving $R^2/I$, [see Eq. (6)], we might expect it to be very different for wood and aluminum bats. The hollow thin-walled construction of an
aluminum bat implies that it has a larger moment of inertia about the longitudinal axis \(I_z\) than a wooden bat of comparable mass and shape. This advantage is partially offset by the disadvantage of having a center of mass farther from the impact point \(\ell b\) is larger), which increases \(r_e\). As a simple exercise, we have investigated two bats with the shape of an R161, one a solid wooden bat and the other a thin-walled aluminum bat. Both bats are 34 in. long and weigh 31.5 oz. The wooden bat has \(I_0=2578\) and \(I_z=18.0\) oz in.\(^2\) and the center of mass 22.7 in. from the knob. The aluminum bat has \(I_0=2985\) and \(I_z=29.3\) oz in.\(^2\) and the center of mass 20.2 in. from the knob. With an impact 6 in. from the barrel end, where the bat diameter is 2.625 in. and \(e_c=0.16\), we have \(e_T=-0.03\) and 0, respectively, for the wooden and aluminum bat. We conclude that, generally speaking, an aluminum bat is marginally more effective in putting backspin on the baseball than a wooden bat of comparable mass and shape.

C. Insights into the scattering process

Besides the obvious practical implications of our result, it is interesting to ask what it teaches us about the scattering process itself. As mentioned, our measured value \(e_T=0\) necessarily implies that \(e_c=0.16\). A value \(e_c<0\) would be obtained if the ball slides on the surface throughout the collision, whereas a value \(e_c>0\) would be obtained if the ball is rolling when it leaves the bat. However, a positive value of \(e_c\) necessarily implies tangential compliance in the ball, the bat, or both. A rigid baseball impacting on a rigid bat without any tangential compliance in the contact region will slide on the bat until the contact point comes to rest, in which case it will enter a rolling mode and will continue to roll with zero tangential velocity as it bounces off the bat.\(^{13}\) However, a real baseball and a real bat can store energy elastically as a result of deformation in directions both perpendicular and parallel to the impact surface. In that case, if tangential velocity is lost temporarily during the collision, then it can be regained from the elastic energy stored in the ball and the bat as a result of tangential deformation in the contact region. The ball will then bounce in an “overspinning” mode with \(r_{o2}>v_{c2}\) or with \(e_c>0\). The details of this process were first established by Maw et al.\(^{14,15}\) The effect is most easily observed in the bounce of a superball,\(^{16}\) which has a tangential coefficient of restitution typically greater than 0.5.\(^{3,4}\) A simple lumped-parameter model for the bounce of a ball with tangential compliance has been developed by Stronge.\(^{17}\)

As mentioned, the signature for continuous sliding throughout the collision is \(e_c<0\). Referring to Fig. 4, data with \(e_c<0\) would lie above or below the dashed line for values of \(\frac{v_{c2}-v_{c1}}{(1+e_A)}v_{c1}\) greater than or less than zero, respectively. Not a single collision satisfies that condition in the present data set, suggesting that \(\mu_k\) is large enough to bring the sliding to a halt. Therefore the scattering data can be used to set a lower limit on \(\mu_k\), which must be at least as large as the ratio of tangential to normal impulse to the center of mass of the ball:

\[
\mu_k \geq \frac{\int F dt}{\int N dt} = \frac{v_{c2}-v_{c1}}{(1+e_A)v_{c1}}.
\]  

(10)

In Fig. 7 values of the right-hand side of Eq. (10) are plotted as a function of the initial ratio of tangential to normal speed. If we use the results derived in Appendix A, it is straightforward to show that these quantities are linearly proportional with a slope equal to \((2/7)(1+e_T)/(1+e_A)\), provided that the initial velocity ratio is below the critical value needed to halt the sliding. Stronge has shown\(^1\) and we confirm with our own formalism that the critical value is \((7/2)\mu_k(1+e_A)/(1+r_e)\), which in our experiment assumes the numerical value 5.6\(\mu_k\). Above the critical value, the impulse ratio should be constant and equal to \(\mu_k\). Given that the data still follow a linear relationship up to an initial velocity ratio of 2.4, corresponding to an impulse ratio of 0.50, we conclude that \(\mu_k \approx 0.50\). If the actual \(\mu_k\) were as small as 0.50, the critical value of the initial velocity ratio would be 2.8, which exceeds the maximum value in our experiment. The lower limit of 0.50 is larger than the lower limit of 0.31 that we measured from oblique collisions of a nonspinning ball with the floor. For that experiment the angle with the horizontal needed to achieve continuous slipping is less than 20°, which is smaller than our minimum angle of 25°. Although no attempt was made to measure the ball-bat directly, our lower limit is consistent with \(\mu_k=0.50\pm0.04\) measured in Ref. 1.

Finally, we remark on our finding that the scattering data are consistent with \(D=0\), implying that the angular momentum of the ball is conserved about the initial contact point. At low enough initial speed, the deformation of the ball will be negligible, so that the ball and bat interact at a point and the angular momentum of the ball is necessarily conserved about that point. Evidently, this condition is satisfied at \(4\) m/s initial speed. It is interesting to speculate whether this condition will continue to be satisfied at the much higher speeds in the game of baseball, where the ball experiences considerable deformation and a significant contact area during the collision. Simple physics considerations suggest that it will not. A ball with topspin incident at an oblique angle will have a larger normal velocity at the leading edge than the trailing edge, resulting in a shift of the line of action of the normal force ahead of the center of mass of the ball \((D>0)\). A
similar shift occurs when brakes are applied to a moving automobile, resulting in a larger normal force on the front wheels than on the back. Such a shift has been observed in high-speed collisions of tennis balls. Whether a comparable shift occurs in high-speed baseball collisions will have to be answered with appropriate experimental data.

V. SUMMARY AND CONCLUSIONS

We have performed a series of experiments in which a baseball is scattered from a bat at an initial speed of about 4 m/s. For the bat that was used in the experiment, we find the horizontal apparent coefficient of restitution $e_T$ is consistent with 0 and inconsistent with the value $-0.14$ that would be expected if the ball is rolling at the end of its impact. These results necessarily imply tangential compliance in the ball, the bat, or both. We further find that the data are consistent with conservation of angular momentum of the ball about the contact point and with a coefficient of sliding friction larger than 0.50. Our results be expected if the ball is rolling at the end of its impact. These results necessarily imply tangential compliance in the ball, the bat, or both. We further find that the data are consistent with conservation of angular momentum of the ball about the contact point and with a coefficient of sliding friction larger than 0.50. Our results suggest that a curveball can be hit with greater backspin than a fastball, but by an amount that is less than would be the case in the absence of tangential compliance. Because our investigations were done at low speed, we must proceed with caution before applying them to the higher speeds that are typical of baseball games.

APPENDIX A: RELATIONSHIP BETWEEN $e_T$ AND $e_x$

We derive for tangential collisions the relation, Eqs. (4) and (6), between $e_T$ and $e_x$. The derivation follows closely that presented in Ref. 8 for the relation between $e_A$ and $e_x$.

We first solve the simple problem involving the collision of two point objects in one dimension. Object A of mass $m$ and velocity $v_1$ is incident on stationary object B of mass $M$. Object A rebounds backward with velocity $v_2$ and object B recoils with velocity $V$. Our sign convention is that $v_1$ is always positive and $v_2$ is positive if the latter is in the opposite direction to $v_1$. The collision is completely determined by conservation of momentum

$$\int F dt = m(v_2 + v_1) = MV,$$

and the coefficient of restitution

$$e = \frac{v_2 + V}{v_1},$$

where $F$ is the magnitude of the force that the two objects exert on each other. We define the apparent coefficient of restitution

$$e_A = \frac{v_2}{v_1},$$

and seek a relation between $e_A$ and $e$. We use Eq. (A3) and write $e$ as

$$e = e_A + \frac{V}{v_1}.$$  

We then use Eq. (A1) to find

$$V = (1 + V/v_1) \frac{m}{M},$$

from which we derive the desired expression

$$e_A = \frac{e - mL}{1 + mL}.$$

We next generalize this derivation for the collision of extended objects, as shown in Figs. 1 and 8. A ball of mass $m$ and radius $r$ is incident obliquely in the $xy$ plane on a stationary bat of mass $M$ and radius $R$ at the impact point. The origin of the coordinate system is at the center of mass of the bat, with the $z$ axis along the longitudinal axis and the $x$ and $y$ axes in the tangential and normal directions, respectively. The impact point $P$ has the coordinates $(0, R, b)$. The ball is incident with angular velocity $\omega_1$ and linear velocity components $v_{x1}$ and $v_{y1}$; it rebounds with angular velocity $\omega_2$ and linear velocity $v_{x2}$ and $v_{y2}$, where the angular velocities are about the $z$ axis. The bat recoils with center of mass (c.m.) velocity components $V_x$ and $V_y$ and with angular velocity about the c.m. with components $\Omega_x$, $\Omega_y$, and $\Omega_z$. Let $v_{p1x}$, $v_{p2x}$, and $V_{px}$ denote the pre- and post-impact velocities of the ball and the post-impact velocity of the bat at the point $P$, respectively. Because we are concerned with tangential collisions, we only consider the $x$ components of these velocities, which are given by

$$v_{p1x} = v_{x1} - r \omega_1,$$

$$v_{p2x} = v_{x2} - r \omega_2,$$

$$V_{px} = V_x + b \Omega_y + R \Omega_z.$$  

From the definitions in Eqs. (1) and (2), we have

$$e_x = \frac{v_{p2x} - V_{px}}{v_{p1x}},$$

and

$$e_T = \frac{v_{p2x}}{v_{p1x}}.$$  

We apply the impulse-momentum expressions to the bat and find

$$\int F dt = MV_x,$$

and

$$b \int F dt = I_0 \Omega_y.$$
\[ R \int F dt = I_0 \Omega_z, \]  
(A9c)

where it has been assumed that the bat is symmetric about the z axis so that \( I_x = I_y = I_0 \). If we combine Eq. (A9) with Eq. (A7), we find

\[ \int F dt = M_{ex} V_{px}, \]  
(A10)

where \( M_{ex} \), the bat effective mass in the x direction, is given by

\[ \frac{1}{M_{ex}} = \frac{1}{M} + \frac{b^2}{I_0} + \frac{R^2}{I_z}. \]  
(A11)

If we apply the impulse-momentum expressions to the ball, we find

\[ \int F dt = -m(v_{x2} - v_{x1}), \]  
(A12a)

\[ r \int F dt - D \int N dt = \alpha m r^2 (\omega_2 - \omega_1). \]  
(A12b)

We note that \( \int N dt = (1 + e_\lambda)mv_{y1} \) and combine Eq. (A12) with Eq. (A7) and find

\[ \int F dt = -m_{ex}(v_{p2x} - v_{p1x}) + \frac{m_{ex}Dv_{y1}(1 + e_\lambda)}{ra}, \]  
(A13)

where \( m_{ex} \), the ball effective mass in the x direction, is given by

\[ m_{ex} = \frac{\alpha}{1 + \alpha} m. \]  
(A14)

If we combine Eqs. (A10) and (A13), we arrive at

\[ -m_{ex}(v_{p2x} - v_{p1x}) + \frac{m_{ex}Dv_{y1}(1 + e_\lambda)}{ra} = M_{ex} V_{px}, \]  
(A15)

which is analogous to the momentum conservation equation for point bodies, Eq. (A1), provided the velocities refer to those at the contact point \( P \) and the masses are effective masses. Following the derivation for point masses, we combine Eq. (A15) with the definitions of \( e_x \) and \( e_T \) to arrive at the result

\[ e_T = e_x - \frac{m_{ex}M_{ex}}{1 + m_{ex}/M_{ex}} + \frac{D}{ra} \left( \frac{r_5}{1 + r_5} \right) v_{y1}(1 + e_\lambda). \]  
(A16)

If we define \( r_5 = m_{ex}/M_{ex} \) and assume that \( \alpha = 2/5 \), then Eq. (A16), along with Eqs. (A11) and (A14), is identical to Eqs. (4) and (6). Our results are equivalent to those used in Ref. 1. We note that Stronge\(^7\) has derived an expression that is equivalent to Eq. (6) for the special case of a bat with zero length, implying \( b=0 \), and \( D=0 \).

**APPENDIX B: COLLISION FORMULAS IN THE LABORATORY REFERENCE FRAME**

Equations (8) and (9) for \( v_{x2} \) and \( \omega_2 \) are valid in the reference frame in which the bat is initially at rest at the impact point. The usual (or laboratory) frame that is relevant for baseball is the one where both the bat and ball initially approach each other. We now derive relations for \( v_{x2} \) and \( \omega_2 \) in the laboratory frame. Our coordinate system is the same as that shown in Fig. 8, where the y axis is normal and the x axis is parallel to the ball-bat contact surface. In this system the initial velocity components of the bat and ball at the impact point are denoted by \( (V_x, V_y) \) and \( (v_{x1} - r_01, v_{y1}) \), respectively, where the usual situation has \( V_x > 0 \) and \( v_{y1} < 0 \). In the bat rest frame, the components of the bat initial velocity at the impact point are therefore \( (v_{x1} - r_01, v_{y1} - V_y) \). If we apply the definitions of \( e_T \) and \( e_x \), the components of the ball velocity after the collision are given by

\[ v_{x2} - r_02 - V_x = e_T(v_{x1} - r_01 - V_x), \]  
(B1a)

\[ v_{y2} - V_y = e_x(v_{y1} - V_y), \]  
(B1b)

which can be rearranged to arrive at

\[ v_{x2} - r_02 = -e_T(v_{x1} - r_01) + (1 + e_T)V_x, \]  
(B2a)

\[ v_{y2} = e_xv_{y1} + (1 + e_x)V_y. \]  
(B2b)

Finally, we combine Eq. (B2a) with the expression for angular momentum conservation, Eq. (7), to find

\[ v_{x2} = v_{x1} \frac{1 - e_T}{1 + e_T} + (r_01 + V_x) \frac{\alpha(1 + e_T)}{1 + \alpha}, \]  
(B3a)

\[ r_02 = r_01 \frac{\alpha - e_T}{1 + \alpha} + (v_{x1} - V_y) \frac{(1 + e_x)}{1 + \alpha}. \]  
(B3b)

Equations (B2) and (B3) are the desired results. Equation (B2b) has appeared in the literature many times.\(^{18,19}\) To our knowledge, this is the first time an explicit formula for \( v_{x2} \) and \( \omega_2 \) in the laboratory frame has been given. Although explicit relations were not given, the earlier works of Ref. 1 and Watts and Baroni\(^2\) are equivalent to ours for the special case \( e_x=0 \), the latter being equivalent to \( e_T=-0.137 \) for our bat. Our relations represent a generalization of their work for arbitrary \( e_x \).

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Hygromeik. The Hygromeik is used to measure relative humidity. Like the hygrometer, it uses wet and dry bulb thermometers. However, it has a nomograph that enables the user to set the wet and dry bulb temperatures and then read off the relative humidity. Loyd’s Hygromeik was patented in 1902. This example, in the Greenslade collection, will also give the absolute humidity and the dew point. It is listed at $17.50 in the 1929 Chicago Apparatus Co. catalogue. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)